modulation analog continuous-wave (CW) (AM) amplitude (linear)
angle phase (PM)
freq (FM)
pulse
digital

4 Amplitude/Linear Modulation

4.1. A sinusoidal carrier signal $A\cos(2\pi f_c t + \phi)$ has three basic parameters: amplitude, frequency, and phase. Varying these parameters in proportion to the baseband signal results in amplitude modulation (AM), frequency[16] modulation (FM), and phase modulation (PM), respectively. Collectively, these techniques are called **continuous-wave (CW) modulation** [13, p 111][3, p 162].

Linear

Definition 4.2. Amplitude modulation is characterized by the fact that the amplitude A of the carrier $A\cos(2\pi f_c t + \phi)$ is varied in proportion to the baseband (message) signal $m(t)$.

- Because the amplitude is time-varying, we may write the modulated carrier as

$$A(t)\cos(2\pi f_c t + \phi)$$

Ex. $A(t) = A_c m(t)$

- Because the amplitude is linearly related to the message signal, this technique is also called **linear modulation**.

4.3. Linear modulations:

(a) **Double-sideband** amplitude modulation

  (i) Double-sideband-suppressed-carrier (DSB-SC or DSSC or simply DSB) modulation

  (ii) Standard amplitude modulation (AM)

(b) Suppressed-sideband amplitude modulation

  (i) Single-sideband modulation (SSB)

  (ii) Vestigial-sideband modulation (VSB)

[16] Technically, the variation of "frequency" is not as straightforward as the description here seems to suggest. For a sinusoidal carrier, a general modulated carrier can be represented mathematically as

$$x(t) = A(t)\cos(2\pi f_c t + \phi(t)).$$

Frequency modulation, as we shall see later, is resulted from letting the time derivative of $\phi(t)$ be linearly related to the modulating signal. [14, p 112]

4.1 Double-sideband suppressed carrier (DSB-SC) modulation

Definition 4.4. In **double-sideband-suppressed-carrier** (**DSB-SC** or DSSC or simply DSB) modulation, the modulated signal is

amplitude scaling factor

$$x(t) = A_c \cos(2\pi f_c t) \times m(t).$$

We have seen that the multiplication by a sinusoid gives two shifted and scaled replicas of the original signal spectrum:

$$X(f) = \frac{A_c}{2} M(f - f_c) + \frac{A_c}{2} M(f + f_c).$$

- When we set $A_c = \sqrt{2}$, we get the "simple" modulator discussed in Example 3.12.

Ex. AM radio $\left.\begin{array}{l} f_c \approx 1\text{ MHz} \\ B \approx 5\text{ kHz} \end{array}\right\} \Rightarrow \frac{f_c}{B} = 200$

- We need $f_c > B$ to avoid spectral overlapping. In practice, $f_c \gg B$.

4.5. Synchronous/coherent detection by the **product demodulator**: The incoming modulated signal is first multiplied with a locally generated sinusoid with the same phase and frequency (from a local oscillator (LO)) and then lowpass-filtered, the filter bandwidth being the same as the message bandwidth B or somewhat larger.

4.6. A DSB-SC modem with no channel impairment is shown in Figure 12.

$y(t) = x(t)$

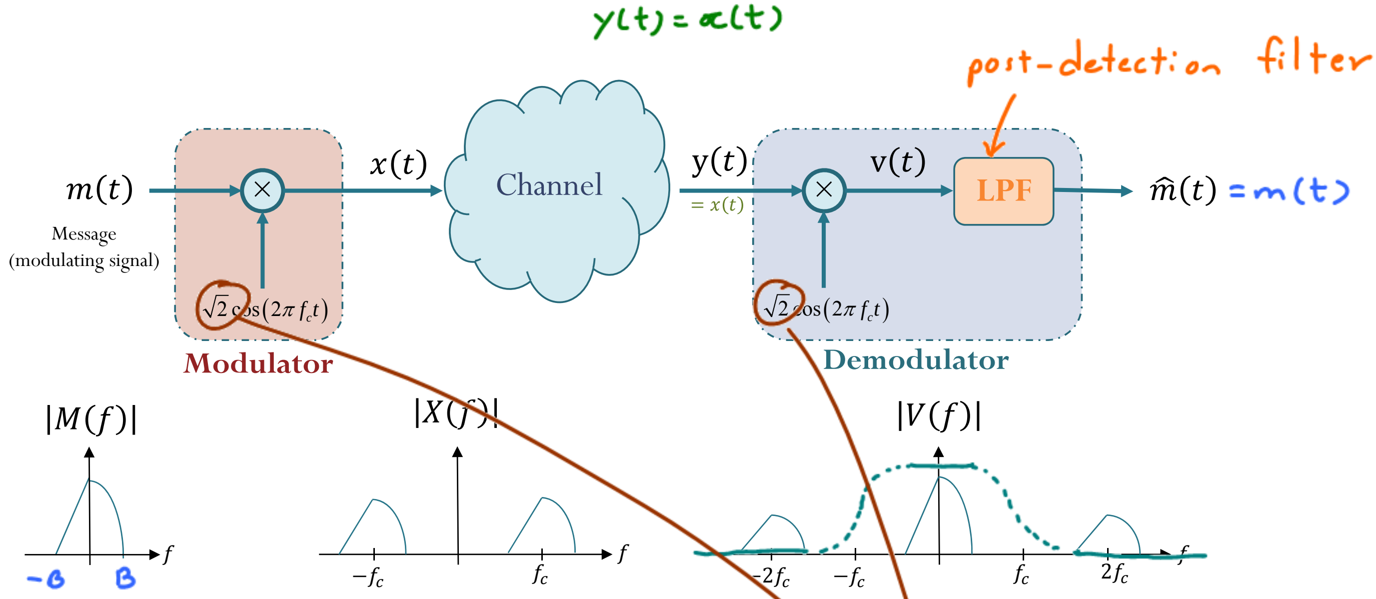


Figure 12: DSB-SC modem with no channel impairment

$$H_{LP}(f) = \begin{cases} 1, & |f| \le B, \\ 0, & |f| \ge 2f_c - B, \\ \text{"any"}, & \text{otherwise.} \end{cases}$$

Once again, recall that $m(t) \times \sqrt{2}\cos(2\pi f_c t) \equiv x(t)$

$$
\begin{aligned}
X(f) &= \sqrt{2}\left(\frac{1}{2}\Big(M(f-f_c)+M(f+f_c)\Big)\right) \\
&= \frac{1}{\sqrt{2}}\left(M(f-f_c)+M(f+f_c)\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
v(t) &= y(t) \times \sqrt{2}\cos(2\pi f_c t) = \sqrt{2}x(t)\cos(2\pi f_c t) \\
V(f) &= \frac{1}{\sqrt{2}}\left(X(f-f_c)+X(f+f_c)\right)
\end{aligned}
$$

$$= \frac{1}{2}\Big(M(\underbrace{f-f_c}-f_c) + M(\underbrace{f-\cancel{f_c}}+\cancel{f_c}) + M(f+\cancel{f_c}-\cancel{f_c}) + M(f+f_c+f_c)\Big)$$

$$= M(f) + \cancel{\frac{1}{2}M(f-2f_c)} + \cancel{\frac{1}{2}M(f+2f_c)}$$

$$\hat{M}(f) = \mathrm{LPF}\{V(f)\} = M(f) + 0 + 0$$

eliminated by LPF

Alternatively, we can work in the time domain and utilize the trig. identity from Example 2.4:

$$
\begin{aligned}
v(t) &= \sqrt{2}x(t)\cos(2\pi f_c t) = \sqrt{2}\left(\sqrt{2}m(t)\cos(2\pi f_c t)\right)\cos(2\pi f_c t) \\
&= 2m(t)\cos^2(2\pi f_c t) = m(t)\left(\cos(2(2\pi f_c t))+1\right) \\
&= m(t) + \cancel{m(t)\cos(2\pi(2f_c)t)}
\end{aligned}
$$

$$\hat{m}(t) = \mathrm{LPF}\{v(t)\} = m(t) + 0$$

Key equation for DSB-SC modem:

$$
\mathrm{LPF}\left\{\underbrace{\left(m(t) \times \sqrt{2}\cos(2\pi f_c t)\right)}_{x(t)} \times \left(\sqrt{2}\cos(2\pi f_c t)\right)\right\} = m(t). \qquad (31)
$$

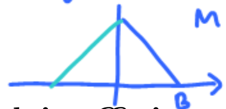
4.7. Implementation issues:

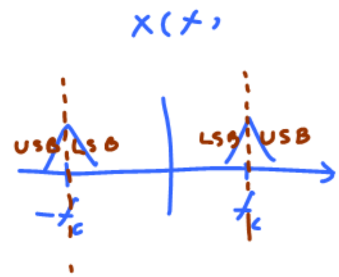
(a) Problem 1: Modulator construction

(b) Problem 2: Synchronization between the two (local) carriers/oscillators

(c) Problem 3: Spectral inefficiency

Recall, $m(t)$ is real-valued and hence its $M(f)$ has conjugate symmetry.





4.8. Spectral **inefficiency/redundancy**: When $m(t)$ is real-valued, its spectrum $M(f)$ has conjugate symmetry. With such message, the corresponding modulated signal's spectrum $X(f)$ will also inherit the symmetry but now centered at f_c (instead of at 0). The portion that lies above f_c is known as the **upper sideband** (USB) and the portion that lies below f_c is known as the **lower sideband** (LSB). Similarly, the spectrum centered at $-f_c$ has upper and lower sidebands. Hence, this is a modulation scheme with **double sidebands**. Both sidebands contain the same (and complete) information about the message.

4.9. Synchronization: Observe that (31) requires that we can generate $\cos(\omega_c t)$ both at the transmitter and receiver. This can be difficult in practice. Suppose that the frequency at the receiver is off, say by Δf, and that the phase is off by θ. The effect of these frequency and phase offsets can be quantified by calculating

$$\cos A \cos B = \frac{1}{2}\left(\underbrace{\cos(A+B)}_{\xrightarrow{\text{LPF}} 0} + \cos(A-B)\right)$$

$$\text{LPF}\left\{\left(m(t)\sqrt{2}\cos(2\pi f_c t)\right)\sqrt{2}\cos(2\pi(f_c+\Delta f)t+\theta)\right\},$$

which gives

$$m(t)\cos(2\pi(\Delta f)t+\theta).$$

When Δf is small, the $\underbrace{\text{"cos" factor}}$ will scale the message

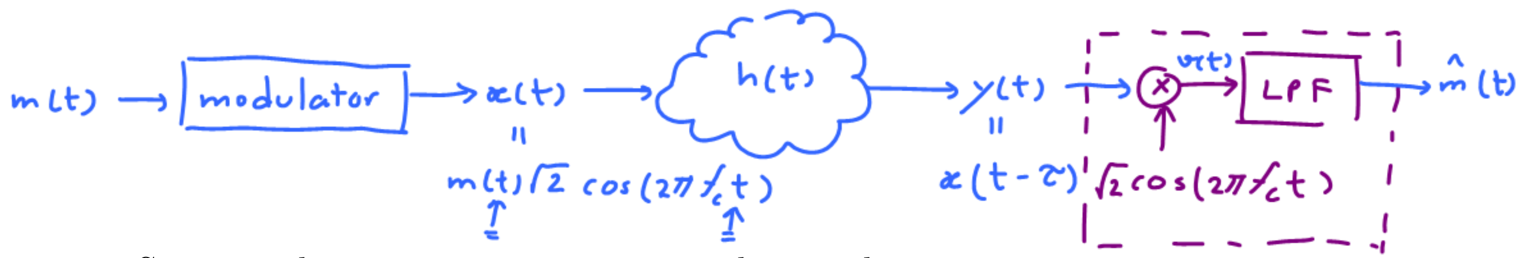
↳ could be near 0 for a while

When $\Delta f = 0$, $m(t)\cos(\theta)$ could $= 0$ at all time when $\theta = 90°, 270°, \ldots$

Of course, we want $\Delta\omega = 0$ and $\theta = 0$; that is the receiver must generate a carrier in phase and frequency synchronism with the ***incoming carrier***.

(propagation time) $\tau = \frac{d}{c}$

4.10. Effect of **time delay**:



Suppose the propagation time is τ, then we have

$$\begin{aligned} y(t) = x(t-\tau) &= \sqrt{2}m(t-\tau)\cos(2\pi f_c(t-\tau)) \\ &= \sqrt{2}m(t-\tau)\cos(2\pi f_c t - 2\pi f_c \tau) \\ &= \sqrt{2}m(t-\tau)\cos(2\pi f_c t - \phi_\tau). \end{aligned}$$

Consequently,

$$\begin{aligned} v(t) &= y(t) \times \sqrt{2}\cos(2\pi f_c t) \\ &= \sqrt{2}m(t-\tau)\cos(\overset{A}{2\pi f_c t - \phi_\tau}) \times \sqrt{2}\cos(\overset{B}{2\pi f_c t}) \\ &= m(t-\tau)\,2\cos(\underset{A}{2\pi f_c t - \phi_\tau})\cos(\underset{B}{2\pi f_c t}). \end{aligned}$$

Applying the product-to-sum formula, we then have

$$v(t) = m(t-\tau)\left(\underbrace{\cos\overset{A+B}{(2\pi(2f_c)t - \phi_\tau)}}_{\text{LPF} \to 0} + \cos\overset{A-B}{(\phi_\tau)}\right).$$

$\hat{m}(t) = \text{LPF}\{v(t)\} = m(t-\tau)\cos(\phi_\tau)$

$\phi_\tau \to 2\pi f_c \tau$

suppose $2\pi f_c \tau = \frac{\pi}{2} + \pi k$, $\tau = \frac{d}{c}$

$\cos(\ ) = 0 \Rightarrow$ bad

$d = \frac{\lambda}{4} + \frac{\lambda}{2}k \Leftarrow 2\cancel{\pi} f_c \frac{d}{c} = \frac{\cancel{\pi}}{2} + \cancel{\pi}k$

In conclusion, we have seen that the principle of the DSB-SC modem is based on a simple key equation (31). However, as mentioned in 4.7, there are several implementation issues that we need to address. Some solutions are provided in the subsections to follow. However, the analysis will require some knowledge of Fourier series which is reviewed in the next subsection.

4.2 Fourier Series "FS"

Definition 4.11. Exponential Fourier series: Let the (real or complex) signal $r(t)$ be a *periodic* signal with period T_0. Suppose the following *Dirichlet* conditions are satisfied

$\frac{1}{T_0} \equiv f_0 \leftarrow$ fundamental freq.

(a) $r(t)$ is absolutely integrable over its period; i.e., $\int_0^{T_0} |r(t)| dt < \infty$.

(b) The number of maxima and minima of $r(t)$ in each period is finite.

(c) The number of discontinuities of $r(t)$ in each period is finite.

Then $r(t)$ can be "expanded" in terms of the complex exponential signals $\left(e^{jn\omega_0 t}\right)_{n=-\infty}^{\infty}$ as

$e^{j2\pi n f_0 t}$ ← Almost the same as the complex-expo in the inverse Fourier transform

$$\tilde{r}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}\right) \qquad (32)$$

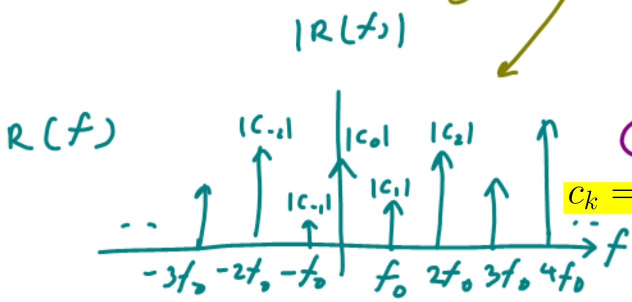
$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$

where

$e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f - f_0)$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0},$$

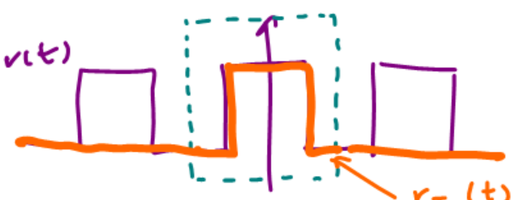
However, there, f can be any real number. Here, the values of f are limited to only multiples of f_0.



$-j2\pi k f_0 t$

$$c_k = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} r(t) e^{-jk\omega_0 t} dt, \qquad (33)$$

So, f is continuous in FT. f is discrete in FS.



$R(f) = \int_{-\infty}^{\infty} r(t) e^{-j2\pi f t} dt$

$r_{T_0}(t) \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} r_{T_0}(t) e^{-j2\pi f t} dt \equiv R_{T_0}(f) = \int_{T_0} r_{T_0}(t) e^{-j2\pi f t} dt$

for some *arbitrary* α. In which case,

$$\tilde{r}(t) = \begin{cases} r(t), & \text{if } r(t) \text{ is continuous at } t \\ \frac{r(t^+)+r(t^-)}{2}, & \text{if } r(t) \text{ is not continuous at } t \end{cases}$$

We give some remarks here.

- The parameter α in the limits of the integration (33) is arbitrary. It can be chosen to simplify computation of the integral. Some references simply write $c_k = \frac{1}{T_0}\int\limits_{T_0} r(t)\, e^{-jk\omega_0 t} dt$ to emphasize that we only need to integrate over one period of the signal; the starting point is not important.

- The coefficients $c_k = \frac{1}{T_0}\int\limits_{T_0} r(t)\, e^{-jk\omega_0 t} dt$ are called the (k^{th}) ***Fourier (series) coefficients*** of (the signal) $r(t)$. These are, in general, complex numbers.
- $c_0 = \frac{1}{T_0}\int\limits_{T_0} r(t)\, dt$ = average or DC value of $r(t)$
- The quantity $f_0 = \frac{1}{T_0}$ is called the ***fundamental frequency*** of the signal $r(t)$. The nth multiple of the fundamental frequency (for positive n's) is called the nth ***harmonic***.
- $c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}$ = the k^{th} ***harmonic component*** of $r(t)$.
  $k = 1 \Rightarrow$ ***fundamental component*** of $r(t)$.

4.12. Getting the Fourier coefficients from the Fourier transform: Consider a restricted version $r_{T_0}(t)$ of $r(t)$ where we only consider $r(t)$ for one specific period. Suppose $r_{T_0}(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} R_{T_0}(f)$. Then,

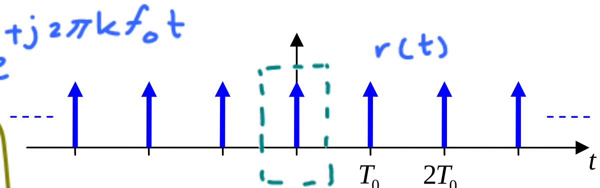
$$c_k = \frac{1}{T_0} R_{T_0}(kf_0).$$

So, ***the Fourier coefficients are simply scaled samples of the Fourier transform***.

Example 4.13. Find the Fourier series expansion for the train of impulses $\delta^{(T_0)}(t) = \sum\limits_{n=-\infty}^{\infty} \delta(t - nT_0)$ drawn in Figure 13.

$c_k = \frac{1}{T_0} R_{T_0}(kf_0)$

$= \sum\limits_{k=-\infty}^{\infty} \frac{1}{T_0} e^{+j2\pi k f_0 t}$

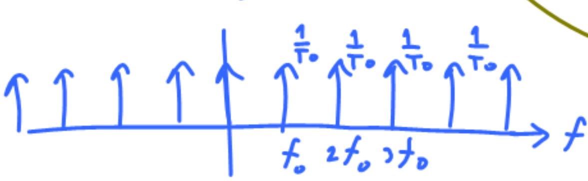


$= \frac{1}{T_0} \times 1 = \frac{1}{T_0}$

Figure 13: Train of impulses

$r_{T_0}(t) = \delta(t) \longrightarrow R_{T_0}(f) \equiv 1$

$\mathcal{F}$



4.14. The Fourier series in Example 4.13 gives an interesting Fourier transform pair:

A special case when $T_0 = 1$ is quite easy to remember:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n) \qquad\qquad X(f) = \sum_{k=-\infty}^{\infty} \delta(f-k)$$

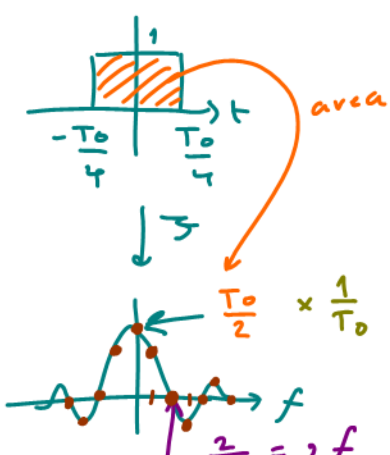
We can use the scaling property of the delta function to generalize the special case:

$$x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$\frac{1}{|a|}\sum_{n} \delta\left(t-\frac{n}{a}\right) = \sum_{n=-\infty}^{\infty} \delta(at-n) \xrightarrow{\mathcal{F}} \frac{1}{|a|}\sum_{k=-\infty}^{\infty} \delta\left(\frac{f}{a}-k\right) = \sum_{k} \delta(f-ka)$$

$$\sum_{n} \delta\left(t-\frac{n}{a}\right) \xrightarrow{\mathcal{F}} |a| \sum_{k} \delta(f-ka) \qquad a = \frac{1}{T_0}$$

Example 4.15. Find the Fourier coefficients of the **square pulse periodic signal** [5, p 57]shown in Figure 14. Note that multiplication by this signal is equivalent to a switching (ON-OFF) operation.



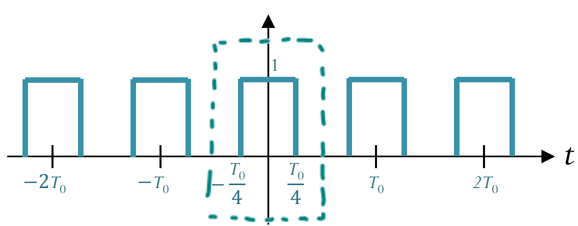


Figure 14: Square pulse periodic signal

$\frac{1}{T_0/2} =$ **4.16.** Parseval's Identity: $P_r = \frac{1}{T_0}\int\limits_{T_0} |r(t)|^2\, dt = \sum\limits_{k=-\infty}^{\infty} |c_k|^2$

Summary:

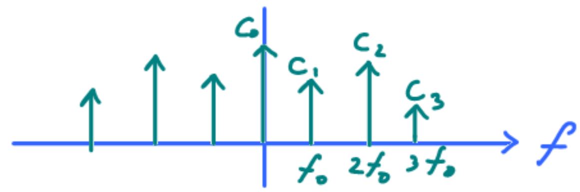
① For periodic signal $r(t)$ with period T_0,

Fourier series expansion: $r(t) \text{ "=" } \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t}$ where $f_0 = \frac{1}{T_0}$

$\uparrow$ fundamental freq.

$\mathcal{F}$ $\qquad$ $\mathcal{F}$

Fourier transform: $R(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_0)$



② The Fourier series coefficients can be found by

(2.1) the formula $\quad c_k = \frac{1}{T_0} \int_{T_0} r(t) e^{-j2\pi kf_0 t} dt$

The integration is done over one period of $r(t)$.

Rely on the fact that you know how to do Fourier transform (and don't want to waste the skill that has already been built) → (2.2) the recipe:

(i) find $r_{T_0}(t)$ ← the restricted version of $r(t)$

use only one period

(ii) find $R_{T_0}(f)$ ← the Fourier transform of $r_{T_0}(t)$

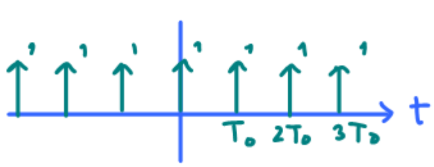
(iii) $c_k = \frac{1}{T_0} R_{T_0}(kf_0)$

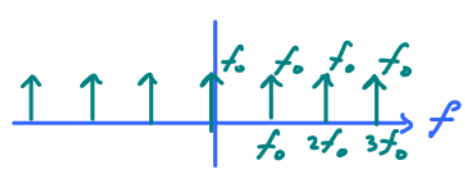
③ Train of impulses

Fourier series expansion

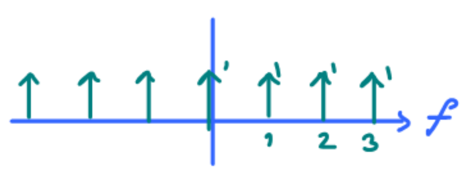
$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{j2\pi kf_0 t} \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \frac{1}{T_0} \delta(f - kf_0)$$

f_0

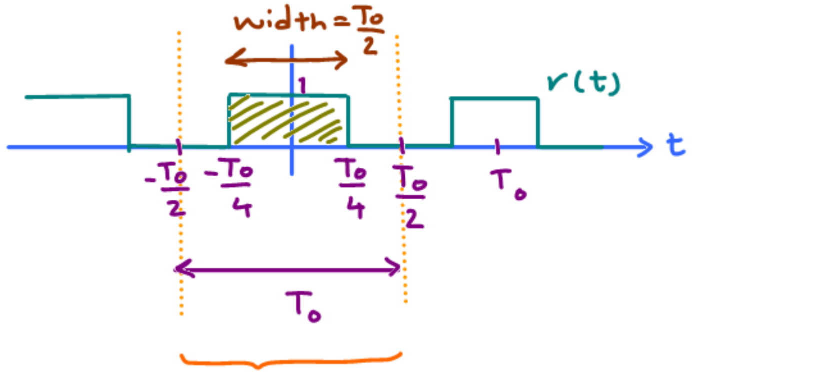




special (and easy-to-remember) case: $T_0 = 1$



④ Square Wave



$r_{T_0}(t)$ uses $r(t)$ only in this region

$$R_{T_0}(f) = \frac{T_0}{2}\,\text{sinc}\left(2\pi \frac{T_0}{4} f\right)$$

area = width × 1 = width = $\frac{T_0}{2}$

$$\frac{1}{\text{width}} = \frac{1}{T_0/2} = \frac{2}{T_0} = 2f_0$$

"period" of the sinc $= 4f_0$

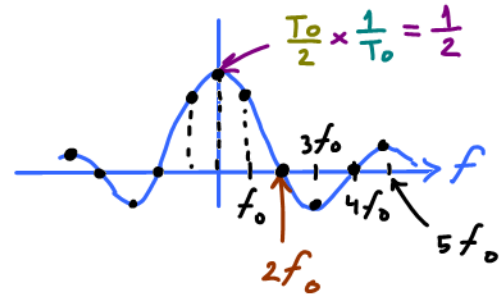
"freq." " " $= \frac{1}{4f_0} = \frac{T_0}{4}$

$$R_{T_0}(kf_0) = \frac{T_0}{2}\,\text{sinc}\left(2\pi \frac{\cancel{T_0}}{4} \times k\cancel{f_0}\right)$$

$$= \frac{T_0}{2}\,\text{sinc}\left(k \times \frac{\pi}{2}\right)$$

$$c_k = \frac{1}{T_0} R_{T_0}(kf_0)$$

$$\frac{T_0}{2} \times \frac{1}{T_0} = \frac{1}{2}$$



$$c_k = \frac{1}{2}\,\text{sinc}\left(k\frac{\pi}{2}\right)$$

$$c_0 = \frac{1}{2}$$

$$c_1 = \frac{1}{2}\,\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{1}{2} \times \frac{1}{\frac{\pi}{2}} = \frac{1}{2} \times \frac{2}{\pi} = c_{-1}$$

$$c_2 = \frac{1}{2}\,\frac{\sin\left(2 \times \frac{\pi}{2}\right)}{2 \times \frac{\pi}{2}} = 0 = c_{-2}$$

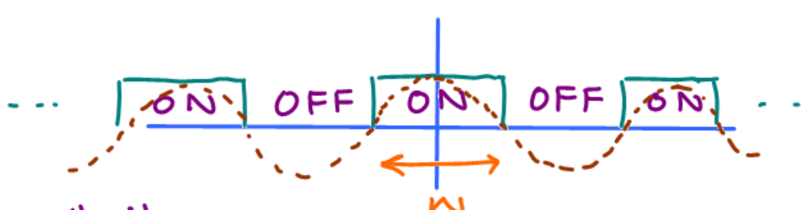
The second harmonic (along with its multiples) is suppressed.

$$c_3 = \frac{1}{2}\,\frac{\sin\left(3 \times \frac{\pi}{2}\right)}{3 \times \frac{\pi}{2}} = \frac{1}{2}\,\frac{-1}{3 \times \frac{\pi}{2}} = -\frac{1}{2} \times \frac{2}{\pi} \times \frac{1}{3}$$

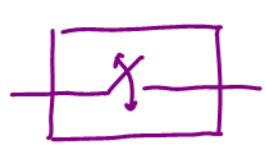
$$= c_{-3}$$

$$c_4 = 0 = c_{-4}$$

4.1 The ON-OFF interpretation



So, we can call it a switching function



4.2 Can express it in the form of a cosine function

$$1\left[\cos(2\pi f_0 t) \geq 0\right]$$

4.3 Duty cycle $\equiv \dfrac{\text{"ON" length}}{\text{period}} = \dfrac{\text{width}}{\text{period}} = \dfrac{w}{T_0}$

In our example, $w = \dfrac{T_0}{2} \Rightarrow$ duty cycle $= \dfrac{T_0/2}{T_0} = \dfrac{1}{2}$

$= 50\%$

When the duty cycle $= \dfrac{1}{n}$, the n^{th} harmonic (c_n) (along with its multiples) is suppressed.

4.4
$$r(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_0 t} + \sum_{k=1}^{\infty} c_{-k} e^{-j2\pi k f_0 t}$$

when $c_k = c_{-k}$

$$= c_0 + \sum_{k=1}^{\infty} c_k \left(e^{j2\pi k f_0 t} + e^{-j2\pi k f_0 t}\right)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k \times 2\cos(2\pi k f_0 t)$$

4.17. Fourier series expansion for real valued function: Suppose $r(t)$ in the previous section is real-valued; that is $r^* = r$. Then, we have $c_{-k} = c_k^*$ and we provide here three alternative ways to represent the Fourier series expansion:

$$\tilde{r}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right) \tag{34}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) \right) + \sum_{k=1}^{\infty} \left(b_k \sin(k\omega_0 t) \right) \tag{35}$$

$$= c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \angle c_k) \tag{36}$$

where the corresponding coefficients are obtained from

$$c_k = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} r(t) e^{-jk\omega_0 t} dt = \frac{1}{2}(a_k - jb_k) \tag{37}$$

$$a_k = 2\mathrm{Re}\{c_k\} = \frac{2}{T_0} \int_{T_0} r(t) \cos(k\omega_0 t)\, dt \tag{38}$$

$$b_k = -2\mathrm{Im}\{c_k\} = \frac{2}{T_0} \int_{T_0} r(t) \sin(k\omega_0 t)\, dt \tag{39}$$

$$2|c_k| = \sqrt{a_k^2 + b_k^2} \tag{40}$$

$$\angle c_k = -\arctan\left(\frac{b_k}{a_k} \right) \tag{41}$$

$$c_0 = \frac{a_0}{2} \tag{42}$$

The Parseval's identity can then be expressed as

$$P_r = \frac{1}{T_0} \int_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2$$

4.18. To go from (34) to (35) and (36), note that when we replace c_{-k} by c_k^*, we have

$$\begin{aligned} c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} &= c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \\ &= c_k e^{jk\omega_0 t} + \left(c_k e^{jk\omega_0 t}\right)^* \\ &= 2\,\mathrm{Re}\left\{c_k e^{jk\omega_0 t}\right\}. \end{aligned}$$

- Expression (36) then follows directly from the phasor concept:

$$\mathrm{Re}\left\{c_k e^{jk\omega_0 t}\right\} = |c_k| \cos\left(k\omega_0 t + \angle c_k\right).$$

- To get (35), substitute c_k by $\mathrm{Re}\{c_k\} + j\,\mathrm{Im}\{c_k\}$ and $e^{jk\omega_0 t}$ by $\cos(k\omega_0 t) + j\sin(k\omega_0 t)$.

Example 4.19. For the train of impulses in Example 4.13,

$$\delta^{(T_0)}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} = \frac{1}{T_0} + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos k\omega_0 t \tag{43}$$

Example 4.20. For the rectangular pulse train in Example 4.15,

$$1[\cos\omega_0 t \geq 0] = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right) \tag{44}$$

Example 4.21. Bipolar square pulse periodic signal [5, p 59]:

$$\mathrm{sgn}(\cos\omega_0 t) = \frac{4}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right)$$

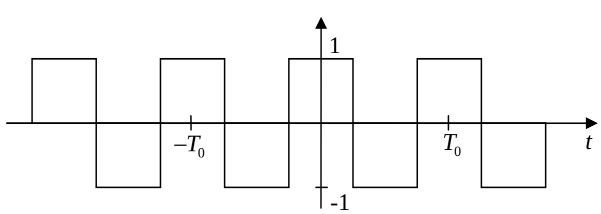


Figure 15: Bipolar square pulse periodic signal

4.3 Classical DSB-SC Modulators

To produce the modulated signal $A_c \cos(2\pi f_c t) m(t)$, we may use the following methods which generate the modulated signal along with other signals which can be eliminated by a bandpass filter restricting frequency contents to around f_c.

4.22. Multiplier Modulators [5, p 184] or **Product Modulator**[3, p 180]: Here modulation is achieved directly by multiplying $m(t)$ by $\cos(2\pi f_c t)$ using an analog multiplier whose output is proportional to the product of two input signals.

- Such a multiplier may be obtained from
    (a) a variable-gain amplifier in which the gain parameter (such as the the β of a transistor) is controlled by one of the signals, say, $m(t)$. When the signal $\cos(2\pi f_c t)$ is applied at the input of this amplifier, the output is then proportional to $m(t)\cos(2\pi f_c t)$.
    (b) two logarithmic and an antilogarithmic amplifiers with outputs proportional to the log and antilog of their inputs, respectively.
        - Key equation:

$$A \times B = e^{(\ln A + \ln B)}.$$

4.23. Square Modulator: When it is easier to build a squarer than a multiplier, use

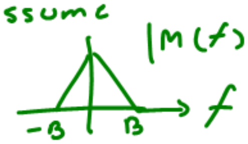
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

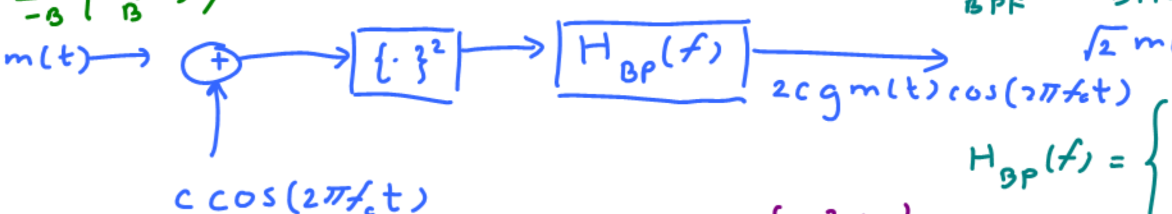
$$\begin{aligned}(m(t) + c\cos(\omega_c t))^2 &= m^2(t) + 2c\,m(t)\cos(\omega_c t) + c^2\cos^2(\omega_c t)\\ &= \cancel{m^2(t)} + 2c\,m(t)\cos(\omega_c t) + \cancel{\frac{c^2}{2}} + \cancel{\frac{c^2}{2}\cos}(2\omega_c t).\end{aligned}$$

$2(2\pi f_c)t$

BPF $\to 0$ BPF $\to 0$ BPF $\to 0$

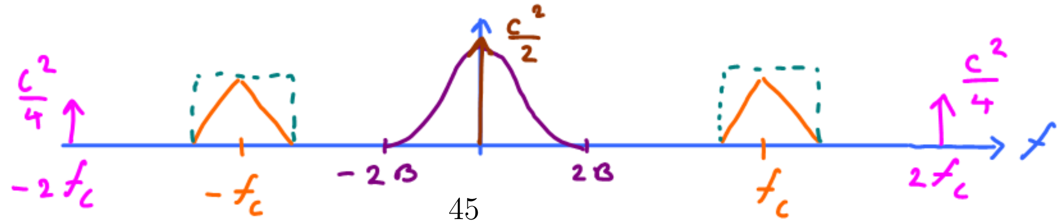
Assume $|M(f)|$





$$H_{BP}(f) = \begin{cases} g, & |f - f_c| \le B \\ g, & |f + f_c| \le B \\ 0, & \text{otherwise} \end{cases}$$

$\mathcal{F}\{m^2(t)\}$



$2B < f_c - B$

$f_c > 3B$

$2cg = \sqrt{2}$

$g = \frac{\sqrt{2}}{2c} = \frac{1}{c\sqrt{2}}$

- Alternative, can use $\left(m(t) + c\cos\left(\frac{\omega_c}{2}t\right)\right)^3$.

4.24. Multiply $m(t)$ by "any" periodic and even signal $r(t)$ whose period is $T_c = \frac{2\pi}{\omega_c}$. Because $r(t)$ is an even function, we know that

$$r\left(t\right) = c_0 + \sum_{k=1}^{\infty} a_k \cos\left(k\omega_c t\right) \text{ for some } c_0, a_1, a_2, \ldots.$$

Therefore,

$$m(t)r\left(t\right) = c_0 m(t) + \sum_{k=1}^{\infty} a_k m(t)\cos\left(k\omega_c t\right).$$

See also [4, p 157]. In general, for this scheme to work, we need

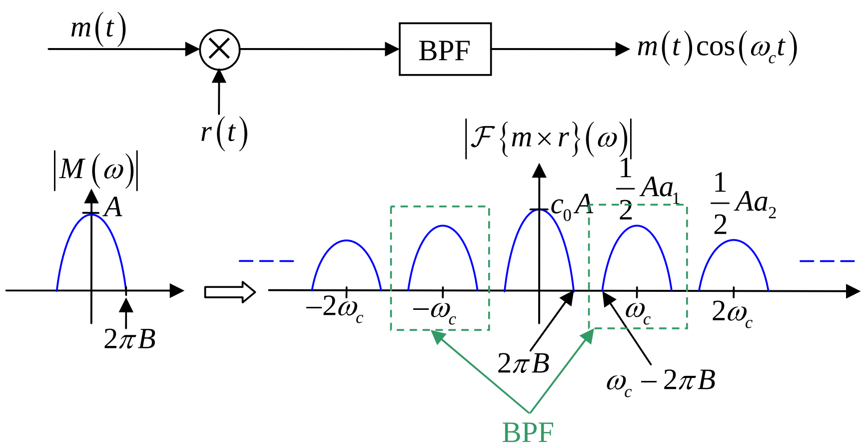


Figure 16: Modulation of $m(t)$ via even and periodic $r(t)$

- $a_1 \neq 0$; that is T_c is the "least" period of r;
- $f_c > 2B$ (to prevent overlapping).

Note that if $r(t)$ is not even, then by (36), the outputted modulated signal is of the form $a_1 m(t)\cos(\omega_c t + \phi_1)$.

4.25. ***Switching modulator***: Set $r(t)$ to be the square pulse train given by (44):

$$
\begin{aligned}
r\,(t) &= 1\,[\cos \omega_0 t \geq 0] \\
&= \frac{1}{2} + \frac{2}{\pi}\left(\cos \omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right).
\end{aligned}
$$

Multiplying this $r(t)$ to the signal $m(t)$ is equivalent to switching $m(t)$ on and off periodically.

It is equivalent to periodically turning the switch on (letting $m(t)$ pass through) for half a period $T_c = \frac{1}{f_c}$.

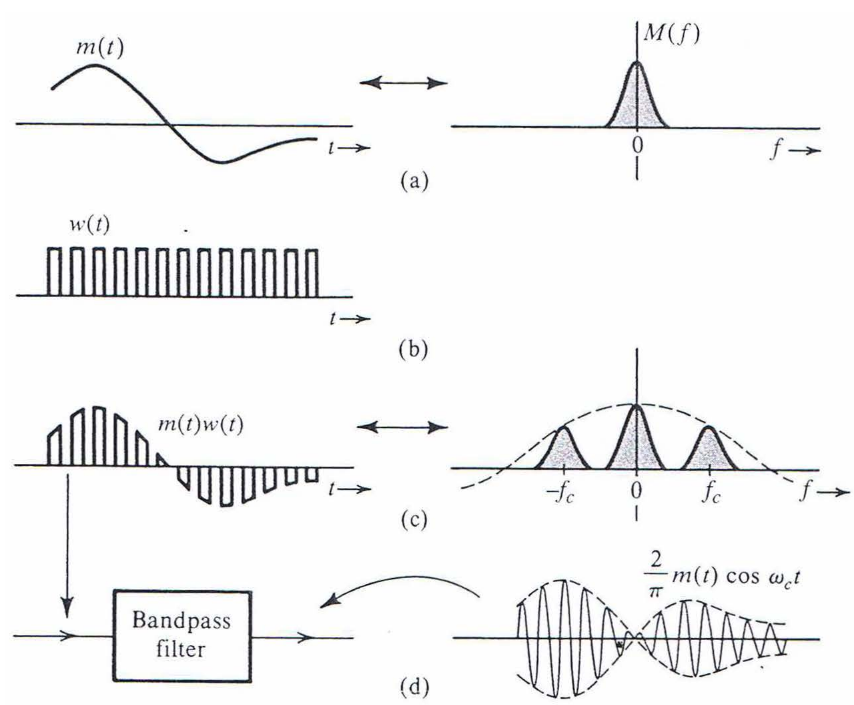


Figure 17: Switching modulator for DSB-SC [4, Figure 4.4].

4.26. ***Switching Demodulator***:

$$
\text{LPF}\{m(t)\cos(\omega_c t) \overbrace{\times\, 1[\cos(\omega_c t) \geq 0]}^{\text{same as switching function}}\} = \frac{1}{\pi} m(t) \tag{45}
$$

[4, p 162]. Note that this technique still requires the switching to be in sync with the incoming cosine as in the basic DSB-SC.